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Generalized Chebyshev polynomials and discrete Schrödinger operators

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Abstract

We investigate a solution of the difference equation

$$tU_n^{A,B}(t) = AU_{n+1}^{A,B}(t) + BU_n^{A,B}(t) + AU_{n-1}^{A,B}(t) \qquad (\star)$$

with the boundary conditions $U_0^{A,B} = I$, $U_{-1}^{A,B} = 0$, where A, B are Hermitian elements of a C^*-algebra. $U_n^{A,B}$ are usually called generalized Chebyshev polynomials of the second kind. The equation $(\star)$ cannot be easily simplified as in the scalar case because A and B do not need to commute. However, we are able to compute the spectrum of the corresponding orthogonality measure which is very important for the investigation of the discrete Schrödinger operator related to $U_n^{A,B}$.

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1. Introduction

The classical Chebyshev polynomials $u_n(x)$ are defined as

$$u_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad \text{where} \quad x = \cos\theta.$$

They are known to satisfy the recurrence formula

$$xu_n(x) = \tfrac{1}{2}u_{n+1}(x) + \tfrac{1}{2}u_{n-1}(x).$$

Moreover they form an orthonormal basis of the space $L^2([-1, 1]; \mu)$ with the weight $\mathrm{d}\mu(x) = \frac{2}{\pi}\sqrt{1-x^2}\,\mathrm{d}x$ supported on the interval $[-1, 1]$.

Now let us denote by $\ell^2(\mathbb{N})$ the classical discrete Hilbert space of all sequences $x = (x_0, x_1, \ldots)$ such that the series $\sum_{n=0}^{\infty} |x_n|^2$ converges, and let $\{e_n\}$ be its canonical orthonormal basis, i.e. $e_n = (0, \ldots, 0, 1, 0, \ldots)$ and 1 appear only in the nth position. On $\ell^2(\mathbb{N})$ we define the shift operator S by the formula $(Sx)_0 = 0$ and $(Sx)_n = x_{n-1}$ for $n \geqslant 1$, i.e. the operator S moves all elements of the sequence x to the right. In other words we have $Se_n = e_{n+1}$.

The classical discrete Schrödinger operator is defined as $J_0 = S + S^*$. Its spectrum is the interval $[-2, 2]$ and its spectral measure w is equal to $\mathrm{d}w\,(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,\mathrm{d}x$.

Now with use of the polynomials $u_n(2x)$ we are able to define Fourier analysis of the Schrödinger operator. Exactly, an operator $\phi\ :\ \ell^2(\mathbb{N}) \mapsto L^2\,([-2, 2]; \mathrm{d}w)$ defined on basis $\{e_n\}$ as $\phi(e_n) = u_n(\frac{1}{2}x)$ is an isometric isomorphism of two Hilbert spaces $\ell^2(\mathbb{N})$ and $L^2\,([-2, 2]; \mathrm{d}w)$. This gives the correspondence $\phi \circ J_0 \circ \phi^{-1} = M_x$, where M_x denotes multiplication by x, i.e. $(M_x f)(x) = xf(x)$. Moreover the spectrum $\sigma(J_0)$ is equal to the support of the weight $w(x)$.

In the following we will need an another tool: moments of the weight w. Denote them by m_n, i.e.

$$m_n = \int_{-2}^{2} x^n w(x)\,\mathrm{d}x. \tag{1}$$

Hence we have

$$\begin{aligned}\langle (J_0)^n e_0 | e_0\rangle &= \langle M_x^n \phi(e_0) | \phi(e_0)\rangle \\ &= \int_{-2}^{2} x^n u_0(\tfrac{1}{2}x)^2\,\mathrm{d}w\,(x) \\ &= m_n.\end{aligned}$$

In the scalar case we have a very simple generalization. Let a and b be real numbers, $a > 0$. Now let us consider an operator $J_{a,b} = a\,J_0 + b\,Id$, where Id denotes the identity operator on $\ell^2(\mathbb{N})$. One can easily see that the corresponding L^2 space is $L^2\big([b - 2a, b + 2a], \mathrm{d}w^{a,b}\big)$, where $\mathrm{d}w^{a,b}\,(x) = \frac{1}{2a\pi}\sqrt{4a^2 - (x - b)^2}\,\mathrm{d}x$.

We can also consider the case $a < 0$ because of simple conjugation

$$M_{(-1)^n} J_{a,b} M_{(-1)^n} = J_{-a,b}$$

where $M_{(-1)^n} e_n = (-1)^n e_n$.

Now a new question arises: what happens if a and b are elements of a more general algebraic structure than $\mathbb{R}$? For example, the matrix case was investigated in [5] and [7]. In the following we present a more general approach.

2. Operator case

2.1. Preliminaries

Let $\mathbb{A}$ be a C^*-algebra. We will write $A \geqslant 0$ (or $A > 0$ respectively) if A is a positive definite (or strictly positive respectively) Hermitian element of $\mathbb{A}$, i.e. $A^* = A$ and $\sigma(A) \subset [0, +\infty)$ (or $\sigma(A) \subset (0, +\infty)$ respectively). In the following the inequality $A \geqslant B$ (or $A > B$ respectively) will be equivalent to $A - B \geqslant 0$ (or $A - B > 0$ respectively) for $A, B \in \mathbb{A}$.

Denote by $\ell^2(\mathbb{A})$ a space of sequences $X = (X_0, X_1, \ldots)$ of elements of $\mathbb{A}$, for which the series $\sum_{n=0}^{\infty} X_n^* X_n$ converges in the norm topology of the C^*-algebra $\mathbb{A}$. We introduce an '$\mathbb{A}$-scalar' product[1] on $\ell^2(\mathbb{A})$:

$$\langle\!\langle X | Y \rangle\!\rangle_{\ell^2} = \sum_{n=0}^{\infty} Y_n^* X_n \in \mathbb{A}.$$

[1] For more details regarding using the name 'scalar' in the case of the spaces $\ell^2(\mathbb{A})$ and $L^2(\Sigma)$ we refer the reader to [2].

Of course, we have $\langle\langle X|X\rangle\rangle_{\ell^2} \geqslant 0$ in the same sense as before.

A system

$$E_n = (0, \dots, 0, I, 0, \dots)$$

where I, which denotes the identity of $\mathbb{A}$, appears only in the nth position, forms an 'orthonormal' basis of $\ell^2(\mathbb{A})$.

In a similar way we can define an L^2-space of square integrable $\mathbb{A}$-valued functions. Let Σ be a positive $\mathbb{A}$-valued Borel measure, i.e. $\Sigma(\Delta)$ is positive definite ($\Sigma(\Delta) \geqslant 0$) for all Borel subsets $\Delta \subset \mathbb{R}$. For $\mathbb{A}$-valued functions $F(x)$ and $G(x)$ we define an '$\mathbb{A}$-scalar' product

$$\langle\langle F|G\rangle\rangle_\Sigma = \int_{\mathbb{R}} F(x)\,\mathrm{d}\Sigma\,(x) G(x)^* \in \mathbb{A}.$$

Now the space $L^2(\Sigma)$ consists of all $\mathbb{A}$-valued functions $F(x)$ for which $\langle\langle F|F\rangle\rangle_\Sigma$ is convergent in the norm topology of the C^*-algebra $\mathbb{A}$.

For more details we refer the reader to [1, 2]. A special case when $\mathbb{A}$ is a matrix algebra $\mathbb{C}^{N\times N}$ which is also of great interest to many authors, is presented in [3, 4, 6, 7].

2.2. *$\mathbb{A}$-valued Schrödinger operator*

Let $A, B \in \ell^2(\mathbb{A})$ be Hermitian and let $J_{A,B}$ be an operator on $\ell^2(\mathbb{A})$ acting as follows:

$$(J_{A,B}X)_0 = BX_0 + AX_1 \qquad (J_{A,B}X)_n = AX_{n-1} + BX_n + AX_{n+1}.$$

Hence $J_{A,B} = B\mathcal{I} + A(\mathcal{S} + \mathcal{S}^*)$, where $\mathcal{S}$ denotes the 'shift' on $\ell^2(\mathbb{A})$ and $\mathcal{I}$—the identity operator.

With $J_{A,B}$ there are associated $\mathbb{A}$-valued Chebyshev polynomials of the second kind $U_n^{A,B}(x)$, i.e. polynomials satisfying the recurrence formula

$$xU_n^{A,B}(x) = AU_{n+1}^{A,B}(x) + BU_n^{A,B}(x) + AU_{n-1}^{A,B}(x).$$

The case where B is a Hermitian matrix and A a positive definite one was fully investigated by Duran in [5].

Denote by M_n the nth moment of $W^{A,B}$:

$$M_n = \int_{\mathbb{R}} x^n\,\mathrm{d}W^{A,B}\,(x) = \langle\langle x^n I|I\rangle\rangle_{W^{A,B}}.$$

Theorem 1. *Let $W^{A,B}$ be an $\mathbb{A}$-valued measure which orthogonalizes polynomials $U_n^{A,B}$, i.e.*

$$\langle\langle U_n^{A,B}|U_m^{A,B}\rangle\rangle_{W^{A,B}} = \int_{\mathbb{R}} U_n^{A,B}(x)\,\mathrm{d}W^{A,B}\,(x) U_m^{A,B}(x)^* = \delta_{n,m} I.$$

Then the moments M_n of the measure $W^{A,B}$ are equal to

$$M_n = \int_{-2}^{2} w(t)(At + B)^n\,\mathrm{d}t. \qquad (2)$$

Moreover

$$\mathrm{supp} W^{A,B} = \bigcup_{t\in[-2,2]} \sigma(At + B).$$

Corollary 2. *The spectrum of $J_{A,B}$ is equal to*

$$\sigma\left(J_{A,B}\right) = \bigcup_{t\in[-2,2]} \sigma(At + B).$$

In the case when $\mathbb{A} = \mathbb{C}^{N\times N}$ and A is invertible, the spectrum $\sigma(J_{A,B})$ consists of at most N non-degenerate intervals of the real line $\mathbb{R}$.

Remark. The corollary covers the results of Duran (cf [5]).

Proof of the corollary. By theorem 2.4, chapter VII, [2] the spectrum of the operator $J_{A,B}$ is equal to the support of the measure $W^{A,B}$. The second statement of the corollary holds because of the continuity of the spectrum. □

Proof of theorem 1. We have

$$\langle\!\langle x^n I | I \rangle\!\rangle_{W^{A,B}} = \langle\!\langle (J_{A,B})^n E_0 | E_0 \rangle\!\rangle_{\ell^2}.$$

Let

$$(B + Ax)^n = \sum_{k=0}^{n} C_{k,n} x^n.$$

Then

$$\begin{aligned}\langle\!\langle (J_{A,B})^n E_0 | E_0 \rangle\!\rangle_{\ell^2} &= \langle\!\langle \left(B\mathcal{I} + A(\mathcal{S} + \mathcal{S}^*)\right)^n E_0 | E_0 \rangle\!\rangle_{\ell^2} \\ &= \sum_{k=0}^{n} C_{k,n} \langle\!\langle (\mathcal{S} + \mathcal{S}^*)^k E_0 | E_0 \rangle\!\rangle_{\ell^2}.\end{aligned}$$

On the other hand $\mathcal{S} E_n = E_{n+1}$, hence the behaviour of $\mathcal{S}$ is the same as that of the operator S. But $\langle\!\langle E_n | E_0 \rangle\!\rangle_{\ell^2} = \delta_{n,0} I = \langle e_n | e_0 \rangle I$, so

$$\langle\!\langle (\mathcal{S} + \mathcal{S}^*)^n E_0 | E_0 \rangle\!\rangle_{\ell^2} = \langle (S + S^*)^n e_0 | e_0 \rangle I.$$

From (1) we have

$$\langle (S + S^*)^n e_0 | e_0 \rangle = m_n = \int_{-2}^{2} t^n w(t)\, \mathrm{d}t.$$

Hence

$$\begin{aligned}M_n &= \int_{-2}^{2} w(t) \sum_{k=0}^{n} C_{k,n} t^n\, \mathrm{d}t \\ &= \int_{-2}^{2} w(t)(At + B)^n\, \mathrm{d}t\end{aligned}$$

which proves the first part of the theorem.

Let

$$\Delta = \bigcup_{t\in[-2,2]} \sigma(At + B).$$

From (2) we have that the equality

$$\int_{\mathbb{R}} p(x)\, \mathrm{d}W^{A,B}(x) = \int_{-2}^{2} w(t) p(At + B)\, \mathrm{d}t \tag{3}$$

holds for every polynomial $p \in \mathbb{C}[x]$. We will now show that $W^{A,B}(F) = 0$ for every compact set F provided $F \subset \mathbb{R} \setminus \Delta$. Let $\varepsilon > 0$. There exists a non-negative polynomial $p \in \mathbb{R}[x]$ such that $p(x) < \varepsilon$ for $x \in \Delta$ and $p(x) > 1$ for $x \in F$. Thus

$$\int_{\mathbb{R}} p(x)\, \mathrm{d}W^{A,B}(x) \geqslant W^{A,B}(F)$$

where the above inequality is in the same sense as was described at the beginning of the section.

On the other hand we have

$$p(At + B) < \varepsilon I$$

for all $t \in [-2, 2]$, because $\sigma(At + B) \subset \Delta$ for such t. Hence

$$\int_{-2}^{2} w(t) p(At + B)\,\mathrm{d}t < \varepsilon I.$$

So

$$W^{A,B}(F) < \varepsilon I$$

for every compact set $F \subset \mathbb{R} \setminus \Delta$ and every $\varepsilon > 0$. This shows that $\operatorname{supp} W^{A,B} \subset \Delta$.

As A and B are Hermitian, Δ is a compact subset of the real line $\mathbb{R}$. Hence $\operatorname{supp} W^{A,B}$ is compact too.

Now by polynomial approximation we have

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}W^{A,B}(x) = \int_{-2}^{2} w(t) f(At + B)\,\mathrm{d}t \tag{4}$$

for every continuous function $f \in C(\Delta)$. Let $G \subset \mathbb{R}$ be an open non-empty subset of the real line $\mathbb{R}$ such that $W^{A,B}(G) = 0$ and let $f \in C(\mathbb{R})$ be a continuous non-negative function such that $f(x) \leqslant 1$ for all $x \in \mathbb{R}$ and $f(x) = 0$ for $x \in \mathbb{R} \setminus G$. Thus

$$0 \leqslant \int_{\mathbb{R}} f(x)\,\mathrm{d}W^{A,B}(x) \leqslant W^{A,B}(G) = 0.$$

Hence

$$\int_{-2}^{2} w(t) f(At + B)\,\mathrm{d}t = 0.$$

But $w(t) f(At + B)$ is continuous and positive definite, so $w(t) f(At + B) = 0$ for $t \in [-2, 2]$. Hence $f(x) = 0$ provided $x \in \sigma(At + B)$ for $t \in [-2, 2]$. Thus $\operatorname{supp} f \subset \mathbb{R} \setminus \Delta$. The above holds for every positive function f satisfying $f(x) \leqslant 1$ and $\operatorname{supp} f \subset G$, so we have $G \subset \mathbb{R} \setminus \Delta$. Hence the support of $W^{A,B}$ is equal to Δ. □

Theorem 3. *Let $\mathbb{A}$ be a subalgebra of the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. Then*

$$W^{A,B}(E) = \int_{-2}^{2} w(t) \mathcal{E}_t(E)\,\mathrm{d}t$$

for every Borel subset $E \subset \mathbb{R}$, where $\mathcal{E}_t$ is the spectral decomposition of $At + B$.

Proof. Let

$$At + B = \int_{\mathbb{R}} \lambda\,\mathrm{d}\mathcal{E}_t(\lambda). \tag{5}$$

Putting (5) into (3) gives

$$\int_{\mathbb{R}} p(x)\,\mathrm{d}W^{A,B}(x) = \int_{-2}^{2} w(t) \int_{\mathbb{R}} p(\lambda)\,\mathrm{d}\mathcal{E}_t(\lambda)\,\mathrm{d}t$$

for every polynomial $p \in \mathbb{C}[x]$. Now by approximation of the characteristic functions we get

$$W^{A,B}(E) = \int_{-2}^{2} w(t) \mathcal{E}_t(E)\,\mathrm{d}t$$

for every Borel subset $E \subset \mathbb{R}$. □

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